A novel approach for solving the three-dimensional sine-Gordon equation

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Abstract

A new way for finding analytical solutions of the three-dimensional sine-Gordon equation is presented. The method is based on the established relation between the solutions of the three-dimensional wave equation and solutions of the three-dimensional sine-Gordon equation. Some examples of the solutions thus obtained are show.

The sine-Gordon equation has been used to describe with a good approximation a number of physical phenomena [1, 2, 3, 4]. The applications can additionally be extended under the condition: dimension of the equation greater than 1. However the methods already known for solving of the one-dimensional sine-Gordon equation cannot be used for finding a solutions of the n-dimensional sine-Gordon equation ($n > 1$).

First of all, we show how a solution of the wave equation

$$\frac{\partial^2 F}{\partial t^2} - \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial z^2} = 0 \qquad F = F\left(x, y, z, t\right) \tag{1}$$

can be used for finding a solution of the three-dimensional sine-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} + \sin u = 0 \qquad u = u\left(x, y, z, t\right). \tag{2}$$

The substitution [5]

$$u = 4\tan^{-1}\sigma \qquad \sigma = \sigma\,(x,y,z,t) \tag{3}$$

leads to the following nonlinear partial differential equation for σ :

$$(1+\sigma^2)[\frac{\partial^2\sigma}{\partial t^2}-\frac{\partial^2\sigma}{\partial x^2}-\frac{\partial^2\sigma}{\partial y^2}-\frac{\partial^2\sigma}{\partial z^2}]-2\sigma[(\frac{\partial\sigma}{\partial t})^2-(\frac{\partial\sigma}{\partial x})^2-(\frac{\partial\sigma}{\partial y})^2-(\frac{\partial\sigma}{\partial z})^2] = \sigma^3-\sigma. \tag{4}$$

One possibility [5] for splitting equation (4) into two is the following:

$$\frac{\partial^2\sigma}{\partial t^2} - \frac{\partial^2\sigma}{\partial x^2} - \frac{\partial^2\sigma}{\partial y^2} - \frac{\partial^2\sigma}{\partial z^2} = -\sigma \tag{5}$$

$$(\frac{\partial\sigma}{\partial t})^2 - (\frac{\partial\sigma}{\partial x})^2 - (\frac{\partial\sigma}{\partial y})^2 - (\frac{\partial\sigma}{\partial z})^2 = -\sigma^2. \tag{6}$$

Proposition *Let F satisfies the wave equation* (1) *and an equation*

$$(\frac{\partial F}{\partial t})^2 - (\frac{\partial F}{\partial x})^2 - (\frac{\partial F}{\partial y})^2 - (\frac{\partial F}{\partial z})^2 = -1; \tag{7}$$

then

$$\sigma = e^F \tag{8}$$

is a solution of the system (5), (6).

Proof. Combining

$$(\frac{\partial F}{\partial t})^2 - (\frac{\partial F}{\partial x})^2 - (\frac{\partial F}{\partial y})^2 - (\frac{\partial F}{\partial z})^2 = [(\frac{\partial\sigma}{\partial t})^2 - (\frac{\partial\sigma}{\partial x})^2 - (\frac{\partial\sigma}{\partial y})^2 - (\frac{\partial\sigma}{\partial z})^2]/\sigma^2$$

and

$$\frac{\partial^2 F}{\partial t^2}-\frac{\partial^2 F}{\partial x^2}-\frac{\partial^2 F}{\partial y^2}-\frac{\partial^2 F}{\partial z^2} = [\frac{\partial^2\sigma}{\partial t^2}-\frac{\partial^2\sigma}{\partial x^2}-\frac{\partial^2\sigma}{\partial y^2}-\frac{\partial^2\sigma}{\partial z^2}]/\sigma-(\frac{\partial F}{\partial t})^2+(\frac{\partial F}{\partial x})^2+(\frac{\partial F}{\partial y})^2+(\frac{\partial F}{\partial z})^2$$

we get the proposition.

Consequently, if $F\,(x,y,z,t)$ is a solution of system (1),(7); then the function $u\,(x,y,z,t)$ is a solution of (2). This are illustrated by following examples.

Example 1. Let

$$F\,(x,y,z,t) = t\sinh\psi + (x\cos\alpha + y\cos\beta + z\cos\gamma)\cosh\psi + C,$$

where $(\cos\alpha, \cos\beta, \cos\gamma)$ is unit vector, and ψ and C are constants. Taking (8) into account, solution (3) becomes

$$u(x, y, z, t) = 4\tan^{-1}[e^{t\sinh\psi + (x\cos\alpha + y\cos\beta + z\cos\gamma)\cosh\psi + C}].$$

We note that this formula denote as three-dimensional version well-know topological soliton of one-dimensional sine-Gordon equation[1, 2, 3, 4].

Example 2. Let

$$F(x, y, z, t) = h(x \pm t) + y\cos\alpha + z\sin\alpha,$$

where h is an arbitrary smooth function and α is constant; then (3) is equivalent to

$$u(x, y, z, t) = 4\tan^{-1}[e^{h(x\pm t) + y\cos\alpha + z\sin\alpha}].$$

We assume that $h = \ln(f)$ and $\alpha = 0$ or $\alpha = \pi$, then

$$u(x, y, t) = 4\tan^{-1}[f(x \pm t)e^{\pm y}]$$

is a solution of the two-dimensional sine-Gordon equation (see[5]).

Example 3. The previous example may be generalized. Suppose a function $F(x, y, z, t)$ has the look:

$$F(x, y, z, t) = h(x\cos\alpha_1 + y\cos\beta_1 + z\cos\gamma_1 \pm t) + x\cos\alpha_2 + y\cos\beta_2 + z\cos\gamma_2,$$

where $(\cos\alpha_1, \cos\beta_1, \cos\gamma_1)$ and $(\cos\alpha_2, \cos\beta_2, \cos\gamma_2)$ are mutually orthgonal unit vectors, and h is an arbitrary smooth function. Using (3) we get new solution

$$u(x, y, z, t) = 4\tan^{-1}[e^{h(x\cos\alpha_1 + y\cos\beta_1 + z\cos\gamma_1 \pm t) + x\cos\alpha_2 + y\cos\beta_2 + z\cos\gamma_2}].$$

Similarly, if $h = \ln(f)$, one obtains the solution

$$u(x, y, z, t) = 4\tan^{-1}[f(x\cos\alpha_1 + y\cos\beta_1 + z\cos\gamma_1 \pm t)e^{x\cos\alpha_2 + y\cos\beta_2 + z\cos\gamma_2}].$$

Example 4. Now suppose the function $F(x, y, z, t)$ has the decomposition into the sum:

$$F(x, y, z, t) = \sum_{i=1}^{k} F_i(x, y, z, t).$$

For example, when $k = 2$,

$$F(x, y, z, t) = h_1(x\cos\alpha_1 + y\cos\beta_1 + z\cos\gamma_1 \pm t) + (x\cos\alpha_2 + y\cos\beta_2 + z\cos\gamma_2)/\sqrt{2}$$

$$+h_2(x\cos\alpha_1 + y\cos\beta_1 + z\cos\gamma_1 \pm t) + (x\cos\alpha_3 + y\cos\beta_3 + z\cos\gamma_3)/\sqrt{2},$$

where h_1, h_2 are arbitrary smooth functions; $(\cos\alpha_1, \cos\beta_1, \cos\gamma_1)$, $(\cos\alpha_2, \cos\beta_2, \cos\gamma_2)$ and $(\cos\alpha_3, \cos\beta_3, \cos\gamma_3)$ are mutually orthgonal unit vectors. Substituting this expression into (8) and the result into (3), we get new solutions of the three-dimensional sine-Gordon equation.

The relations (1) and (7) can easily be checked by a direct calculation.

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